# Coulomb-energy radii from isobaric mass parabolas



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# Coulomb-energy radii from isobaric mass parabolas

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**Abstract.** The mass difference for mirror nuclides is expressed in terms of the curvature $2C_2$ of the corresponding isobaric mass parabola and the (fractional) 'atomic number' $Z_*$ corresponding to its minimum. With the Coulomb energy of a nucleus taken as $\gamma_A Z(Z-1)/A^{1/3}$, it is shown that

$$C_2(A-2Z_*) = \beta + \gamma_A(A^{2/3} - A^{-1/3})$$

where $\beta$ represents the hydrogen–neutron mass difference. The relationship is valid for both odd and even values of $A$. The availability of empirical values of $C_2$ and $Z_*$ permits the evaluation of $\gamma_A$ and thence the 'Coulomb-energy radius' $R_c = r_A A^{1/3}$ where $r_A \equiv \frac{3}{5}e^2/\gamma_A$. Consideration of a 'quantal' formula for the Coulomb energy of a nucleus shows that the 'quantal' values of $r_A$ are slightly smaller than the 'classical' values but the difference becomes negligible for large values of $A$. Although there are significant variations in $r_A$, the general trend is to smaller values of $r_A$ as $A$ gets larger, e.g. $r_A = 1{\cdot}47$ fm for $16 \leqslant A \leqslant 74$ whereas $r_A = 1{\cdot}30$ fm for $120 \leqslant A \leqslant 198$. Values of $r_A$ corresponding to almost all values of $A$ from 16 to 249 are given graphically and estimates of the error are included.

## 1. Introduction

The Coulomb energy of a nucleus with a charge $Ze$ that is uniformly distributed throughout a sphere of radius $R$ is commonly taken as $\frac{3}{5}Z(Z-1)e^2/R$ (Bethe and Bacher 1936, Mayer and Jensen 1955, p. 7). For certain nuclei the Coulomb-energy difference is a measurable quantity and the formula is used to determine nuclear radii. Empirically, it is found that $R/A^{1/3}$ roughly approximates to a constant (Blatt and Weisskopf 1952, p. 15) where $A$ is the mass number. Accordingly, for the present, the Coulomb energy of a nucleus is expressed in the form

$$ {}^A_Z E_c = \frac{\gamma_A Z(Z-1)}{A^{1/3}} \tag{1}$$

and the 'Coulomb-energy radius' (Evans 1955, p. 32) is taken as $R_c = r_A A^{1/3}$ where $r_A \equiv \frac{3}{5}e^2/\gamma_A$. It should be noted that the parameter $\gamma_A$ can be regarded as allowing for the dependence of the Coulomb energy on such factors as the shape of the nucleus and the charge distribution (Swiatecki 1964, p. 58). Consequently, $R_c$ represents an 'equivalent' value corresponding to a spherical nucleus with a uniform charge density and the parameter $r_A$ may serve to indicate variations in the shape of the nucleus (Siemens and Bethe 1967) and/or departures from uniform charge density.

Two isobars ($A$ = constant) that have the proton number $Z$ of one equal to the neutron number $A$–$Z$ of the other are called 'mirror nuclides'. Previous determinations of Coulomb-energy radii from the energy difference for mirror nuclides have been restricted to small values of $A$ (e.g. Elton 1961, p. 56), since mirror nuclides are not 'observed' for $A > 43$. However, in the present paper, it is shown that the mirror-nuclide method can be extended to include all values of $A$, if one employs an 'extrapolation' of the corresponding isobaric mass parabola. The availability of 'Coulomb-energy radii' for the complete range of $A$ values should be helpful in a study of the variation of the Coulomb energy with mass number.

## 2. The mirror-nuclide mass difference

The mass of an atom, relative to $^{12}C$, can be expressed in the form ${}^A_Z M = Au + {}^A_Z\Delta$ where ${}^A_Z\Delta$ is called the 'mass excess' (Fleury and de Boer 1962). Assuming that the 'nuclidic components' are protons, neutrons and electrons, one finds that the mass of a stable atom is less than the sum of the masses of the 'components'. The energy equivalent of the difference is referred to as 'binding energy' and can be expressed in the form

$$ {}^{A}_{Z}E = \alpha A + \beta Z - {}^{A}_{Z}\Delta \qquad \text{(in MeV)} $$

where the values of $\alpha$ and $\beta$ are related to the masses of the 'nuclidic components'. Alternatively, a 'theoretical approach', e.g. Bethe–Weizsäcker equation (Bethe and Bacher 1936), suggests the form

$$ {}^{A}_{Z}E = {}^{A}_{Z}E_{\mathrm{n}} - {}^{A}_{Z}E_{\mathrm{c}} + {}_{Z}E_{\mathrm{e}} $$

where ${}^{A}_{Z}E_{\mathrm{n}}$ represents 'specific' nuclear terms and ${}_{Z}E_{\mathrm{e}}$ represents the 'electronic' binding energy. Thus neglecting the term ${}_{Z}E_{\mathrm{e}}$ which is several orders of magnitude smaller than ${}^{A}_{Z}E_{\mathrm{c}}$ we obtain the relationship

$$ {}^{A}_{Z}\Delta \text{ (in MeV)} = \alpha A + \beta Z - {}^{A}_{Z}E_{\mathrm{n}} + {}^{A}_{Z}E_{\mathrm{c}}. \qquad (2) $$

It is generally assumed that mirror nuclides corresponding to a particular $A$ have a common value for ${}^{A}_{Z}E_{\mathrm{n}}$ (Mayer and Jensen 1955, p. 27). Moreover, mirror nuclides with $A$ odd have $Z = \frac{1}{2}(A \pm 1)$. It therefore follows from equations (1) and (2) that the mirror-nuclide mass difference is given by the relationship

$$ \delta M_0 \equiv {}_{\frac{1}{2}(A+1)}{}^{A}\Delta - {}_{\frac{1}{2}(A-1)}{}^{A}\Delta = \beta + \gamma_A \frac{A-1}{A^{1/3}} \qquad \text{(in MeV).} \qquad (3) $$

The value of $\beta$ corresponding to the 'hydrogen–neutron' hypothesis is $-0{\cdot}783$ MeV. Thus, when $\delta M_0$ is known, equation (3) yields a value for $\gamma_A$ and thence the 'Coulomb-energy radius' of the nucleus.

## 3. The mirror-nuclide mass difference from isobaric mass parabolas

Empirically, it is found that for isobars of a given 'class' of nuclide (e.g. even $A$, even $Z$) a plot of the mass excess ${}^{A}_{Z}\Delta$ against $Z$ is parabolic to a high degree of approximation. The two parabolas associated with even-$A$ nuclides are identical in form but have a relative displacement $\epsilon$ along their common axis (Bethe and Bacher 1936, Bohr and Wheeler 1939). The situation for odd-$A$ isobars is similar, only the displacement $\epsilon$ is much smaller (Glueckhauf 1948) and in most cases is less than $0{\cdot}3$ MeV (Dewdney 1963). Accordingly, the mass excess can be expressed in the form

$$ {}^{A}_{Z}\Delta \text{ (in MeV)} = C_0 + C_1 Z + C_2 Z^2 \qquad (4) $$

where $C_0$, $C_1$ and $C_2$ are functions of $A$ alone, with the qualification that $C_0$ depends also on the 'class' of nuclide. It follows that the common abscissa of the minima of the isobaric mass parabolas is $-(C_1/C_2) \equiv Z_*$. In general, $Z_*$ has a fractional value. Early calculations of $C_2$ and $Z_*$ were necessarily based on scanty experimental data. However, with the empirical data now available, statistical determinations of the parabolic parameters are possible. Values of $C_2$ and $Z_*$, corresponding to values of $A$ from 16 to 249, have been determined by Dewdney (1963) employing total $\beta$-decay energies and by Hillman (1964, pp. 67–73) using nuclidic masses.

A graphical representation of the mirror-nuclide mass difference in terms of the 'constants' of the corresponding isobaric mass parabola is given in figure 1. Introducing the value $Z_0 \equiv \frac{1}{2}A$, it follows that mirror nuclides with $A$ odd have $Z = Z_0 \pm \frac{1}{2}$. Thus from equation (4) we have that $\delta M_0 = C_1 + 2C_2 Z_0$ if $\epsilon = 0$. Therefore, with

$$ I_* \equiv A - 2Z_* = 2(Z_0 - Z_*) $$

and

$$ C_1 + 2C_2 Z_* = 0 $$

it follows that $\delta M_0 \equiv C_2 I_*$. Substituting in equation (3) we have the relationship

$$ C_2 I_* = \beta + \gamma_A \frac{A-1}{A^{1/3}} \qquad (5) $$

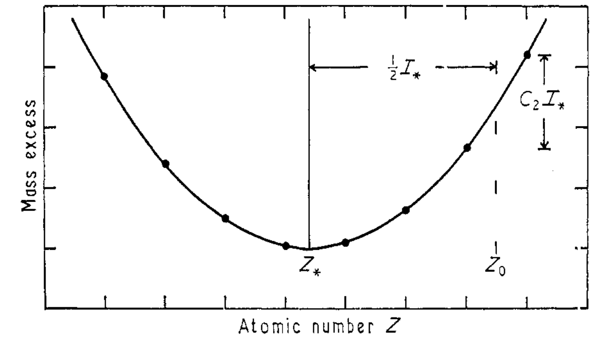


Figure 1. Isobaric mass parabola ${}^{A}_{Z}\Delta = C_0 + C_1 Z + C_2 Z^2$ and the mirror-nuclide mass difference $\delta M_0$ which equals $C_2(A-2Z_*) \equiv C_2 I_*$.

which is valid for odd values of $A$ if $\epsilon$ is negligible with respect to $C_2I_*$. On the other hand, mirror nuclides with $A$ even have $Z = Z_0 \pm 1$ where $Z_0$ is an integer; consequently the mirror nuclides correspond to adjacent points on the same mass parabola and their mass difference $\delta M_e$ is independent of $\epsilon$. It follows from equation (4) that $\delta M_e = 2C_2I_*$. However, from equations (1) and (2) it follows that

$$\delta M_e = 2\beta + 2\gamma_A \frac{A-1}{A^{1/3}}.$$

Therefore, equation (5) is also applicable when $A$ is even. Values of $\gamma_A$ and $r_A$ that are obtained from equation (5) will be referred to as 'classical' values.

## 4. Classical values of $\gamma_A$ and $r_A$

Equation (5) indicates that for $\gamma_A$ equal to a constant, a plot of the empirical values of $C_2I_*$ against $(A^{2/3} - A^{-1/3})$ should be linear with the slope equal to $\gamma_A$ and the intercept equal to $\beta$. On the other hand, if $\gamma_A$ has a range of values, the points should be scattered over a wedge shaped area, with the tip of the wedge corresponding to the common intercept $\beta$. A plot of $C_2I_*$ against $(A^{2/3} - A^{-1/3})$ that employs values of $C_2$ and $I_*$ based on Dewdney's (1963) analysis is shown in figure 2. The scatter for the set of points suggests that $\gamma_A$ has a range of values.

Assuming $\beta = -0{\cdot}783$ MeV, equation (5) has been used to obtain values of $\gamma_A$ and $r_A$. For odd and even values of $A$ from 16 to 198, the mean value of $\gamma_A$ is 0·62 MeV and the mean value of $r_A$ is 1·45 fm. A graphical representation of the values of $r_A$ is given in figure 3; a least-squares fit to a straight line for $16 \leqslant A \leqslant 198$ indicates a trend to smaller values of $r_A$ as $A$ gets larger.

## 5. A quantal relationship for the mirror-nuclide mass difference

Since one might question the appropriateness of equation (1) in dealing with the domain of the nucleus, consideration is now given to the use of a 'quantal' formula. Elton (1961, pp. 52–3) outlines the development of a 'quantal 'relationship for the Coulomb energy of a nucleus which has the form

$$E_c = \tfrac{3}{5} e^2 \frac{Z^2 - KZ^{4/3}}{R}$$

where $K$ is an unspecified constant. An equivalent relationship given by Bethe and Bacher (1936), in effect, assigns the value 0·767 to $K$. Accordingly, the 'quantal' expression is taken as

$$ {}^{A}_{Z}E_{cq} = \gamma_A \frac{Z^2 - 0{\cdot}767 Z^{4/3}}{A^{1/3}}. \tag{6}$$

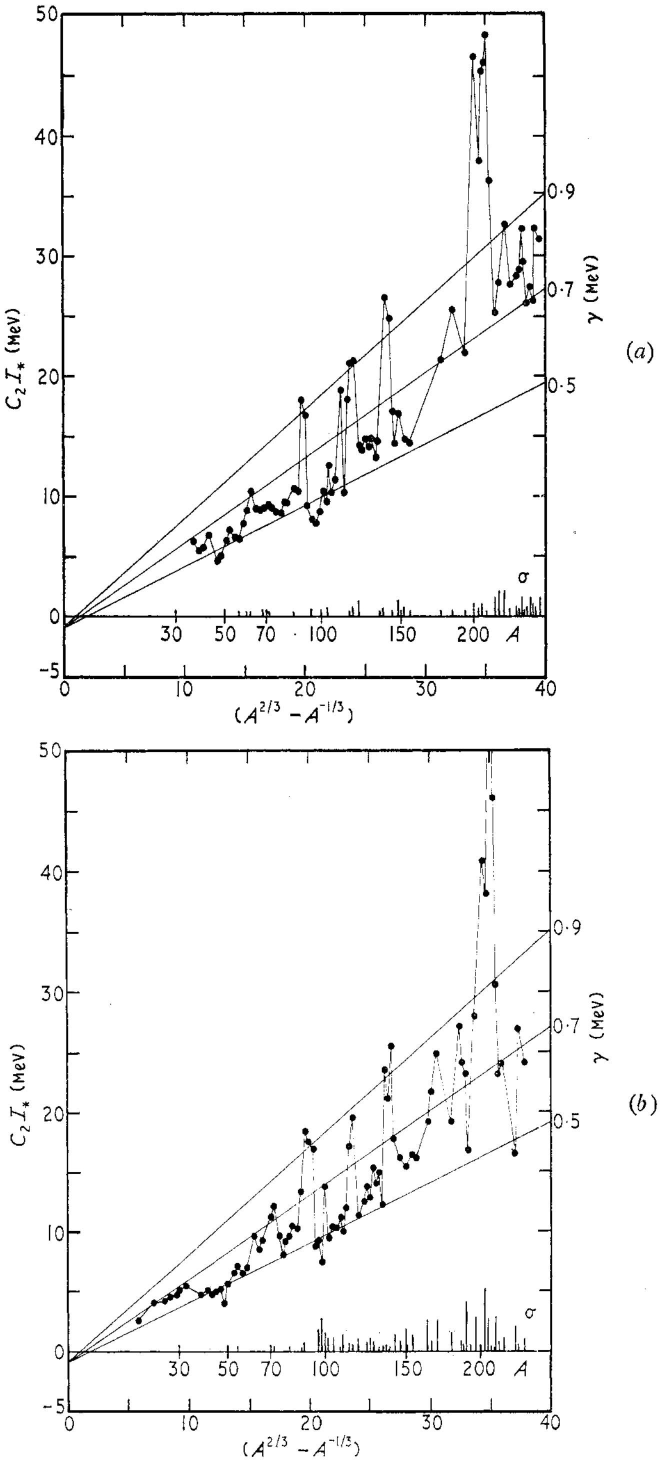


Figure 2. Values of $C_2I_*$ which equals the mirror-nuclide mass difference when $A$ is odd and one half the same when $A$ is even, plotted against $(A^{2/3}-A^{-1/3})$, showing the variation in the 'Coulomb-energy coefficient' $\gamma$ as given by the relationship $C_2I_* = \beta+\gamma(A^{2/3}-A^{-1/3})$ where $A$ is the mass number and $\beta = -0{\cdot}783$ MeV; (*a*) odd $A$ values, (*b*) even $A$ values. Values of $C_2$ and $I_*$ are from Dewdney (1963). The error in $C_2I_*$ is shown only if greater than 0·25 MeV.

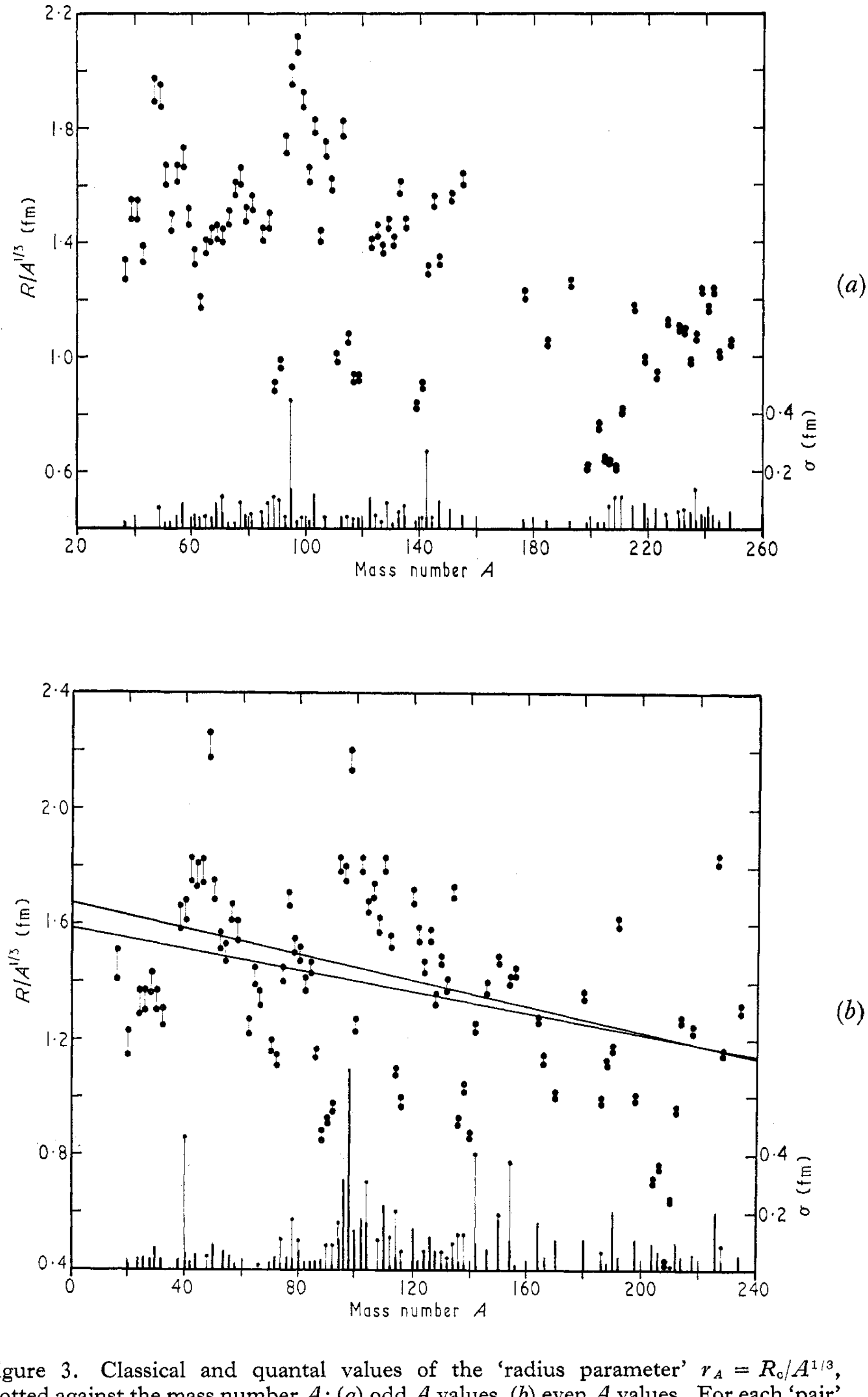


Figure 3. Classical and quantal values of the 'radius parameter' $r_A = R_c/A^{1/3}$, plotted against the mass number $A$; (*a*) odd $A$ values, (*b*) even $A$ values. For each 'pair' of points, the upper point represents the classical value of $r_A$. A least-squares representation of the classical values of $r_A$ for both odd and even values of $A$ up to 198 is given by the upper straight line; similarly, the lower straight line represents the quantal values of $r_A$. The 'internal error' in $r_A$ (heavy line) is shown only if greater than 0·01 fm; the 'external error' (dot) is shown only if greater than the 'internal error'.

It follows that for mirror nuclides with $A$ odd, the Coulomb energy can be expressed in the form

$$^{A}E_{\mathrm{cq}} = \gamma_A \frac{(A\pm 1)^2 - 2^{2/3}(0{\cdot}767)(A\pm 1)^{4/3}}{4A^{1/3}}.$$

Consequently, to a high degree of approximation,

$$\delta(^{A}E_{\mathrm{cq}}) = \gamma_A\{A^{2/3} - \tfrac{2}{3}(0{\cdot}767)2^{2/3}\}.$$

The 'quantal' counterpart of equation (5) is therefore

$$C_2 I_* = \beta + \gamma_A(A^{2/3} - 0{\cdot}812). \qquad (7)$$

Furthermore, recalling that mirror nuclides with $A$ even have $2Z = A\pm 2$, it is readily seen that equation (7) is also applicable when $A$ is even.

## 6. Quantal values of $r_A$

Assuming $\beta = -0{\cdot}783$ MeV, equation (7) has been used to obtain 'quantal' values of $\gamma_A$ and $r_A$; the mean value† of $\gamma_A$ is 0·64 MeV and the mean value† of $r_A$ is 1·41 fm. A plot of the 'quantal' values of $r_A$ against $A$ is given in figure 3; the least-squares straight line indicates a trend to smaller values of $r_A$ as $A$ gets larger. It should be noted that the 'quantal' values of $r_A$ are contingent on the value assigned to the 'exchange constant' (0·767) in equation (6). However, the effect of the 'exchange constant' on the value of $r_A$ diminishes as $A$ increases ($0{\cdot}812 \ll A^{2/3}$). Thus we note that for $16 \leqslant A \leqslant 74$ the value of $\bar{r}_A$ is 1·47 fm whereas for $120 \leqslant A \leqslant 198$ the value of $\bar{r}_A$ is 1·30 fm. Significantly, the latter value is closer to that given by other 'charge sensitive' methods, e.g. electron scattering (Hofstadter 1956) 1·20–1·25 fm, proton scattering (Glassgold 1958) 1·25 fm, and $\mu$-mesonic atoms (Fitch and Rainwater 1953) 1·19 fm.

## 7. Discussion

Equations (5) and (7) indicate that the 'classical' and 'quantal' values of $\gamma_A$ should be essentially the same for large values of $A$. Thus, it is of interest to note that the intersection of the two least-squares lines in figure 3 corresponds to $r_A \simeq 1{\cdot}16$ fm. Furthermore, it is apparent from figure 3 that certain changes in $A$ produce significant changes in $r_A$ irrespective of the relationship employed for the Coulomb energy. In this regard, it should be noted that the 'extreme' values of $r_A$ cannot be attributed, in general, to extremely large experimental error or marked deviation from 'parabolic systematics'. In Dewdney's (1963) analysis a measure of the former is given by the 'internal' error and a measure of the latter is given by the 'external' error.

Figure 2 shows estimates of the 'internal' error in $C_2 I_*$ that are greater than 0·25 MeV which corresponds to the radius of a 'marker' dot. The 'estimates' are standard deviations $\sigma$ for the product $C_2 I_*$ and are based on Dewdney's values of $\sigma$ for $C_2$ and $\sigma$ for $Z_*$. It is worth noting, however, that $C_2$ and $Z_*$ are determined from the same data and have a positive correlation; therefore the 'uncertainty' in $C_2 I_*$ should be somewhat less than that indicated. Figure 3 shows the 'internal' errors in $r_A$ that are greater than 0·01 fm and also the available 'external' errors that are greater than the corresponding 'internal' error. The average 'internal' error in $r_A$ for 156 $A$ values is 0·047 fm, with 34 values less than 0·01 fm and 18 values greater than 0·1 fm. On the other hand, the average 'external' error in $r_A$ for 68 $A$ values is 0·090 fm, with 6 values less than 0·01 fm and 6 values greater than 0·2 fm. The ratio of the 'external' error to the 'internal' error is, on the average, close to 3, if one excludes the 9 cases where the ratio is greater than 10; the latter are, in the main, associated with 'magic numbers' as pointed out by Dewdney (1963). Nevertheless, in 11 cases out of 68 the 'external' error is less than the 'internal' error.

† ($16 \leqslant A \leqslant 198$).

It is evident that the present determination of Coulomb-energy radii for values of $A$ greater than 43 depends on an 'extrapolation' of the empirical mass parabola to $Z = \frac{1}{2}A$. An examination of the 'external' errors indicates that in many cases the parabolic systematics is quite exact and warrants some degree of extrapolation. It should be noted, however, that the quantity of interest is $C_2 I_*$ which is the 'slope' of the parabola at $Z = \frac{1}{2}A$ and the extrapolation is therefore based on a 'linear' function of Z. Accordingly the 'error' in the (estimated) slope of the parabola at $Z = \frac{1}{2}A$ is approximately $I_*$ times the 'error' in $C_2$ where the latter is determined from a least-squares fit of known values to the 'linear' function of $Z$ (Dewdney 1963).

It is noteworthy that the determination of Coulomb-energy radii from isobaric mass parabolas permits a much more extensive investigation ($16 \leqslant A \leqslant 249$) of Coulomb energies than is possible from the consideration of only naturally occurring mirror-nuclides ($A \leqslant 43$). Furthermore, the investigation reveals significant variations in the 'radius parameter' $R_c/A^{1/3}$ which could be the result of non-uniform charge density or deviations from sphericity for the nucleus.

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## References

Bethe, H. A., and Bacher, R. F., 1936, *Rev. Mod. Phys.*, **8**, 82–229.
Blatt, J. M., and Weisskopf, V. F., 1952, *Theoretical Nuclear Physics* (New York: John Wiley).
Bohr, N., and Wheeler, J. A., 1939, *Phys. Rev.*, **56**, 426–50.
Dewdney, J. W., 1963, *Nucl. Phys.*, **43**, 303–15.
Elton, L. R. B., 1961, *Nuclear Sizes* (Oxford: Clarendon Press).
Evans, R. D., 1955, *The Atomic Nucleus* (New York: McGraw-Hill).
Fitch, V. L., and Rainwater, J., 1953, *Phys. Rev.*, **92**, 789–800.
Fleury, P., and de Boer, J., 1962, *Phys. Today*, **15**, 20–30.
Glassgold, A. E., 1958, *Rev. Mod. Phys.*, **30**, 419–23.
Gleuckhauf, E., 1948, *Proc. Phys. Soc.*, **61**, 25–33.
Hillman, M., 1964, *Proc. 2nd Int. Conf. on Nuclidic Masses*, Ed. W. H. Johnson, Jr. (Vienna: Springer-Verlag).
Hofstadter, R., 1956, *Rev. Mod. Phys.*, **28**, 214–54.
Mayer, M. G., and Jensen, J. H. D., 1955, *Elementary Theory of Nuclear Shell Structure* (New York: John Wiley).
Siemens, P. J., and Bethe, H. A., 1967, *Phys. Rev. Lett.*, **18**, 704–6.
Swiatecki, W. J., 1964, *Proc. 2nd Int. Conf. on Nuclidic Masses*, Ed. W. H. Johnson, Jr. (Vienna: Springer-Verlag).